Elemi függvények deriváltja
(Derivácie elementárnych funkcií)

A konstans függvény

$$(c)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{c-c}{x-x_0} = \lim_{x\to x_0}\frac{0}{x-x_0} = 0$$

Hatványfüggvények

$$(x)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x-x_0}{x-x_0} = 1$$

$$(x^2)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x+x_0)}{x-x_0} = \lim_{x\to x_0}(x+x_0) = x_0 + x_0 = 2x_0$$

$$(x^3)' = \lim_{x\to x_0}\frac{x^3-x_0^3}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x^2+xx_0+x_0^2)}{x-x_0} = \lim_{x\to x_0}(x^2+xx_0+x_0^2) = x_0^2 + x_0^2 + x_0^2 = 3x_0^2$$

$$(x^4)' = \lim_{x\to x_0}\frac{x^4-x_0^4}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x^3+x^2x_0+xx_0^2+x_0^3)}{x-x_0} = \lim_{x\to x_0}(x^3+x^2x_0+xx_0^2+x_0^3) = 4x_0^3$$

$$(x^n)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x^n-x_0^n}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x^{n-1}+x^{n-2}x_0+x^{n-3}x_0^2+\cdots+xx_0^{n-2}+x_0^{n-1})}{x-x_0} =$$
$$= \lim_{x\to x_0}(x^{n-1}+x^{n-2}x_0+x^{n-3}x_0^2+\cdots+xx_0^{n-2}+x_0^{n-1}) = n.x_0^{n-1}$$

$$\left(\frac{1}{x^n}\right)' = (x^{-n})' = -n.x^{-n-1} = -n.x^{-(n+1)} = \frac{-n}{x^{n+1}}$$

$$\left(\sqrt[n]{x}\right)' = \left(x^{\frac{1}{n}}\right)' = \frac{1}{n}x^{\frac{1}{n}-1} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{\sqrt[n]{x^{1-n}}}{n} =$$
$$= \frac{1}{n}x^{-\frac{n-1}{n}} = \frac{1}{n.x^{\frac{n-1}{n}}} = \frac{1}{n.\sqrt[n]{x^{n-1}}}$$

példa:

$(3)' = 0$
$(-7)' = 0$
$(-2{,}897)' = 0$
$\left(\frac{101}{13}\right)' = 0$
$\left(\sqrt{11}\right)' = 0$
$(\pi)' = 0$

$(x)' = 1$
$(2x)' = 2.(x)' = 2.1 = 2$
$(11x)' = 11.(x)' = 11.1 = 11$
$(-29x)' = -29(x)' = -29$
$\left(\frac{5}{9}x\right)' = \frac{5}{9}(x)' = \frac{5}{9}$
$\left(\sqrt{3}x\right)' = \sqrt{3}(x)' = \sqrt{3}$

$(x^2)' = 2x$
$(5x^2)' = 5.(x^2)' = 5.2x = 10x$
$(34x^2)' = 34(x^2)' = 34.2x = 68x$
$(-7x^2)' = -7.2x = -14x$
$\left(-\frac{13}{12}x^2\right)' = -\frac{13}{12}.2x = -\frac{13}{6}x$

$(x^3)' = 3x^2$
$(4x^3)' = 4.(x^3)' = 4.3x^2 = 12x^2$
$(22x^3)' = 22.3x^2 = 66x^2$
$(-10x^3)' = -10.3x^2 = -30x^2$

$(x^4)' = 4x^3$
$(x^6)' = 6x^5$

$(x^{14})' = 14x^{13}$

$(x^{501})' = 501x^{500}$

$(4x^7 - 3x^6 + 5x^4)' = (4x^7)' - (3x^6)' + (5x^4)' = 4.(x^7)' - 3.(x^6)' + 5.(x^4)' = 4.7x^6 - 3.6x^5 + 5.4x^3 =$
$= 28x^6 - 18x^5 + 20x^3$

$(2x^{11} - 6x^9 - 3x^5 + 88)' = (2x^{11})' - (6x^9)' - (3x^5)' + (88)' = 2(x^{11})' - 6(x^9)' - 3(x^5)' + 0 =$
$= 22x^{10} - 54x^8 - 15x^4$

$(x^{-7})' = -7x^{-8} = -\frac{7}{x^8}$

$(3x^{-24})' = 3.(x^{-24})' = 3.(-24)x^{-25} = -72x^{-25} = -\frac{72}{x^{25}}$

$(-2x^{-98})' = -2.(x^{-98})' = -2.(-98)x^{-99} = 196x^{-99} = \frac{196}{x^{99}}$

$\left(\frac{1}{x}\right)' = (x^{-1})' = -1x^{-2} = -\frac{1}{x^2}$

$\left(\frac{5}{x}\right)' = (5x^{-1})' = 5.(x^{-1})' = 5.(-1).x^{-2} = -5x^{-2} = -\frac{5}{x^2}$

$\left(\frac{3}{x^2}\right)' = (3x^{-2})' = 3.(-2)x^{-3} = -\frac{6}{x^3}$

$\left(\frac{-8}{x^5}\right)' = (-8x^{-5})' = -8.(-5)x^{-6} = \frac{40}{x^6}$

$\left(\frac{-12}{x^7}\right)' = (-12x^{-7})' = -12.(-7)x^{-8} = \frac{84}{x^8}$

$\left(\frac{2}{x^{13}}\right)' = (2x^{-13})' = 2.(-13)x^{-14} = -\frac{26}{x^{14}}$

$\left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

$\left(\sqrt[3]{x}\right)' = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$

$\left(\sqrt[7]{x}\right)' = \left(x^{\frac{1}{7}}\right)' = \frac{1}{7}x^{-\frac{6}{7}} = \frac{1}{7x^{\frac{6}{7}}} = \frac{1}{7\sqrt[7]{x^6}}$

$\left(\sqrt[5]{x^8}\right)' = \left(x^{\frac{8}{5}}\right)' = \frac{8}{5}x^{\frac{3}{5}} = \frac{8}{5}\sqrt[5]{x^3}$

$\left(\sqrt[11]{x^3}\right)' = \left(x^{\frac{3}{11}}\right)' = \frac{3}{11}x^{-\frac{8}{11}} = \frac{3}{11x^{\frac{8}{11}}} = \frac{3}{11\sqrt[11]{x^8}}$

$\left(7\sqrt[6]{x^{11}}\right)' = \left(7x^{\frac{11}{6}}\right)' = 7.\frac{11}{6}x^{\frac{5}{6}} = \frac{77}{6}\sqrt[6]{x^5}$

$\left(-\frac{2}{3}\sqrt[7]{x^5}\right)' = \left(-\frac{2}{3}x^{\frac{5}{7}}\right)' = -\frac{2}{3}.\frac{5}{7}x^{-\frac{2}{7}} = -\frac{10}{21x^{\frac{2}{7}}} = -\frac{10}{21\sqrt[7]{x^2}}$

$\left(\frac{1}{\sqrt{x}}\right)' = \left(\frac{1}{x^{\frac{1}{2}}}\right)' = \left(x^{-\frac{1}{2}}\right)' = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}} = -\frac{1}{2\sqrt{x^3}}$

$\left(\frac{1}{\sqrt[3]{x}}\right)' = \left(\frac{1}{x^{\frac{1}{3}}}\right)' = \left(x^{-\frac{1}{3}}\right)' = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3x^{\frac{4}{3}}} = -\frac{1}{3\sqrt[3]{x^4}}$

$\left(\frac{1}{\sqrt[12]{x}}\right)' = \left(\frac{1}{x^{\frac{1}{12}}}\right)' = \left(x^{-\frac{1}{12}}\right)' = -\frac{1}{12}x^{-\frac{13}{12}} = -\frac{1}{12x^{\frac{13}{12}}} = -\frac{1}{12\sqrt[12]{x^{13}}}$

$\left(\frac{1}{\sqrt[4]{x^7}}\right)' = \left(\frac{1}{x^{\frac{7}{4}}}\right)' = \left(x^{-\frac{7}{4}}\right)' = -\frac{7}{4}x^{-\frac{11}{4}} = -\frac{7}{4x^{\frac{11}{4}}} = -\frac{7}{4\sqrt[4]{x^{11}}}$

$\left(\frac{1}{\sqrt[13]{x^6}}\right)' = \left(\frac{1}{x^{\frac{6}{13}}}\right)' = \left(x^{-\frac{6}{13}}\right)' = -\frac{6}{13}x^{-\frac{19}{13}} = -\frac{6}{13x^{\frac{19}{13}}} = -\frac{6}{13\sqrt[13]{x^{19}}}$

$\left(\frac{3}{2\sqrt[5]{x^6}}\right)' = \left(\frac{3}{2x^{\frac{6}{5}}}\right)' = \left(\frac{3}{2}x^{-\frac{6}{5}}\right)' = \frac{3}{2}\left(-\frac{6}{5}\right)x^{-\frac{11}{5}} = -\frac{9}{5x^{\frac{11}{5}}} = -\frac{9}{5\sqrt[5]{x^{11}}}$

$\left(\frac{-5}{\sqrt[9]{x^8}}\right)' = \left(\frac{-5}{x^{\frac{8}{9}}}\right)' = \left(-5x^{-\frac{8}{9}}\right)' = -5\left(-\frac{8}{9}\right)x^{-\frac{17}{9}} = \frac{40}{9x^{\frac{17}{9}}} = \frac{40}{9\sqrt[9]{x^{17}}}$

Szögfüggvények

$(\sin x)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\sin x-\sin x_0}{x-x_0} = \lim_{x\to x_0}\frac{2\cos\frac{x+x_0}{2}.\sin\frac{x-x_0}{2}}{x-x_0} = \lim_{x\to x_0}\frac{2\cos\frac{x+x_0}{2}.\sin\frac{x-x_0}{2}}{2.\frac{x-x_0}{2}} =$

$= \lim_{x\to x_0}\cos\frac{x+x_0}{2} = \cos\frac{2x_0}{2} = \cos x_0$

$(\cos x)' = \lim_{x\to x_0}\frac{\cos x-\cos x_0}{x-x_0} = \lim_{x\to x_0}\frac{-2\sin\frac{x+x_0}{2}.\sin\frac{x-x_0}{2}}{x-x_0} = \lim_{x\to x_0}\frac{-2\sin\frac{x+x_0}{2}.\sin\frac{x-x_0}{2}}{2.\frac{x-x_0}{2}} = \lim_{x\to x_0} -\sin\frac{x+x_0}{2} =$

$= -\sin\frac{2x_0}{2} = -\sin x_0$

$(\text{tg } x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'.\cos x-\sin x.(\cos x)'}{(\cos x)^2} = \frac{\cos x.\cos x-\sin x.(-\sin x)}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$(\text{cotg } x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)'.\sin x-\cos x.(\sin x)'}{(\sin x)^2} = \frac{-\sin x.\sin x-\cos x.\cos x}{\sin^2 x} = \frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$

Az összetett függvény (zložená funkcia)

Egy függvény összetett, ha az x helyén egy másik függvényt találunk, mint argumentumot. Lehet többszörösen összetett függvény is.

pl. $\sin 2x$; $\sqrt{x-3}$; 2^{3x-7}; $\log x^3$; $\frac{1}{x^2-7x+3}$; $\text{tg}\sqrt{e^{x+3}}$

Az összetett függvények több függvényből (komponens) állnak. A belső függvény mindig az, amit legelőször kell kiszámolnunk, ha ki szeretnénk számolni a függvényértéket egy konkrét pontban.

$\sin 2x$ az $x = 27°$ pontban
először: $2.27° = 54°$
utána: $\sin 54° = 0{,}809$

$\sqrt{x-3}$ az $x = 14$ pontban
először: $14 - 3 = 11$
utána: $\sqrt{11} = 3{,}317$

2^{3x-7} az $x = 2$ pontban
először: $3.2 - 7 = 6 - 7 = -1$
utána: $2^{-1} = 0{,}5$

$\log x^3$ az $x = 4$ pontban
először: $4^3 = 64$
utána: $\log 64 = 1{,}806\,2$

$\frac{1}{x^2-7x+3}$ az $x = 3$ pontban
először: $3^2 - 7.3 + 3 = 9 - 21 + 3 = -9$
utána: $\frac{1}{-9} = -0{,}1\bar{1}$

$\text{tg}\sqrt{e^{x+3}}$ az $x = 5$ pontban
először: $5 + 3 = 8$
utána: $e^8 = 2\,980{,}96$
később: $\sqrt{2\,980{,}96} = 54{,}60$
és végül: $\text{tg } 54{,}60 = 2{,}506$ (radiánban számítva)

T. $(f\circ g)'(x) = \{f[g(x)]\}' = f'[g(x)].g'(x)$

Az összetett függvényt úgy deriváljuk, hogy először lederiváljuk a külső függvényt a belső függvény, mint pont helyén, és megszorozzuk a belső függvény deriváltjával.

példa:

$(\sin 2x)' = \cos 2x.(2x)' = \cos 2x.2 = 2.\cos 2x$

$\left(\sqrt{x-3}\right)' = \left((x-3)^{\frac{1}{2}}\right)' = \frac{1}{2}(x-3)^{-\frac{1}{2}}.(x-3)' = \frac{1}{2\sqrt{x-3}}.1 = \frac{1}{2\sqrt{x-3}}$

$(2^{3x-7})' = 2^{3x-7}.\ln 2.(3x-7)' = 2^{3x-7}.\ln 2.3 = 3.2^{3x-7}.\ln 2$

$(\log x^3)' = \frac{1}{x^3.\ln 10}.(x^3)' = \frac{1}{x^3.\ln 10}.3x^2 = \frac{3}{x.\ln 10}$

$$\left(\frac{1}{x^2-7x+3}\right)' = ((x^2-7x+3)^{-1})' = -1.(x^2-7x+3)^{-2}.(x^2-7x+3)' = \frac{-1}{(x^2-7x+3)^2}.(2x-7) = \frac{7-2x}{(x^2-7x+3)^2}$$

$$\left(\text{tg}\sqrt{e^{x+3}}\right)' = \frac{1}{\cos^2\sqrt{e^{x+3}}}.\left((e^{x+3})^{\frac{1}{2}}\right)' = \frac{1}{\cos^2\sqrt{e^{x+3}}}.\frac{1}{2}(e^{x+3})^{-\frac{1}{2}}.(e^{x+3})' = \frac{1}{2.\sqrt{e^{x+3}}.\cos^2\sqrt{e^{x+3}}}.e^{x+3}.(x+3)' =$$

$$= \frac{e^{x+3}}{2.\sqrt{e^{x+3}}.\cos^2\sqrt{e^{x+3}}}.1 = \frac{e^{x+3}}{2.\sqrt{e^{x+3}}.\cos^2\sqrt{e^{x+3}}}$$

Exponenciális függvények

$$(e^x)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{e^x-e^{x_0}}{x-x_0} = \lim_{x\to x_0}\frac{e^{x_0}\left(\frac{e^x}{e^{x_0}}-1\right)}{x-x_0} = \lim_{x\to x_0}\frac{e^{x_0}(e^{x-x_0}-1)}{x-x_0} = \lim_{x\to x_0} e^{x_0} = e^{x_0}$$

$$(a^x)' = (e^{x.\ln a})' = e^{x.\ln a}.(x.\ln a)' = a^x.1.\ln a = a^x.\ln a$$

Logaritmusos függvények

$$(\ln x)' = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\ln x-\ln x_0}{x-x_0} = \lim_{x\to x_0}\frac{\ln\frac{x}{x_0}}{x_0\left(\frac{x}{x_0}-1\right)} = \lim_{x\to x_0}\frac{1}{x_0} = \frac{1}{x_0}$$

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}.(\ln x)' = \frac{1}{x.\ln a}$$

Hiperbolikus függvények

$\text{sh } x = \frac{e^x-e^{-x}}{2}$ $\qquad$ $\text{ch } x = \frac{e^x+e^{-x}}{2}$

$\text{th } x = \frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{\text{sh } x}{\text{ch } x}$ $\qquad$ $\text{cth } x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{\text{ch } x}{\text{sh } x}$

Összefoglalás

$(c)' = 0$	$(x^n)' = n.x^{n-1}$
$\left(\frac{1}{x^n}\right)' = -\frac{n}{x^{n+1}}$	$\left(\sqrt[n]{x}\right)' = \frac{1}{n.\sqrt[n]{x^{n-1}}}$
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\text{tg } x)' = \frac{1}{\cos^2 x}$	$(\text{cotg } x)' = \frac{-1}{\sin^2 x}$
$(\text{arc sin } x)' = \frac{1}{\sqrt{1-x^2}}$	$(\text{arc cos } x)' = \frac{-1}{\sqrt{1-x^2}}$
$(\text{arc tg } x)' = \frac{1}{1+x^2}$	$(\text{arc cotg } x)' = \frac{-1}{1+x^2}$
$(e^x)' = e^x$	$(a^x)' = a^x.\ln a$
$(\ln x)' = \frac{1}{x}$	$(\log_a x)' = \frac{1}{x.\ln a}$
$(\text{sh } x)' = \text{ch } x$	$(\text{ch } x)' = \text{sh } x$
$(\text{th } x)' = \frac{1}{\text{ch}^2 x}$	$(\text{cth } x)' = \frac{-1}{\text{sh}^2 x}$
$(\text{ar sh } x)' = \frac{1}{\sqrt{1+x^2}}$	$(\text{ar ch } x)' = \frac{1}{\sqrt{x^2-1}}$
$(\text{ar th } x)' = \frac{1}{1-x^2}$	$(\text{ar cth } x)' = \frac{-1}{1-x^2}$

példa:

$$(x.\sin x)' = (x)'.\sin x + x.(\sin x)' = 1.\sin x + x.\cos x = \sin x + x.\cos x$$

$$(x^2.\text{tg } x)' = (x^2)'.\text{tg } x + x^2.(\text{tg } x)' = 2x.\text{tg } x + x^2.\frac{1}{\cos^2 x}$$

$$(x^3.2^x)' = (x^3)'.2^x + x^3.(2^x)' = 3x^2.2^x + x^3.2^x.\ln 2$$

$$\left(\frac{\ln x}{x^2}\right)' = \frac{(\ln x)'.x^2-\ln x.(x^2)'}{(x^2)^2} = \frac{\frac{1}{x}.x^2-\ln x.2x}{x^4} = \frac{x-2x.\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$$

$$\left(\frac{\sqrt[3]{x^4}}{e^x}\right)' = \frac{\left(x^{\frac{4}{3}}\right)'.e^x-\sqrt[3]{x^4}.(e^x)'}{(e^x)^2} = = \frac{\frac{4}{3}.x^{\frac{1}{3}}.e^x-\sqrt[3]{x^4}.e^x}{(e^x)^2} = \frac{\frac{4}{3}.\sqrt[3]{x}.e^x-\sqrt[3]{x^4}.e^x}{(e^x)^2} = \frac{\frac{4.\sqrt[3]{x}.e^x-3.\sqrt[3]{x^4}.e^x}{3}}{(e^x)^2} = \frac{e^x\left(4.\sqrt[3]{x}-3.\sqrt[3]{x^4}\right)}{3.(e^x)^2} =$$

$$= \frac{4.\sqrt[3]{x}-3.\sqrt[3]{x^4}}{3.e^x}$$

$$(x^x)' = (e^{x.\ln x})' = e^{x.\ln x}.(x.\ln x)' = x^x.((x)'.\ln x + x.(\ln x)') = x^x.\left(1.\ln x + x.\frac{1}{x}\right) = x^x.(\ln x + 1)$$